Explorations of Conway’s Sylver Coinage Game
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Definition
Sylver Coinage is a game in which two players, A and B, alternately name positive integers that are not the sum of nonnegative multiples of previously named integers. The person that is forced to name 1 loses.

Example
Here is an example game between A and B.
A: \( (5) = (5, 10, 15, \ldots) \)
B: \( (4) = (4, 5, 8, 9, 10, 12, \ldots) \)
A: \( (5, 11) = (4, 5, 8, \ldots) \)
B: \( (5, 11, 6, 7, 2) = (2, \ldots) \)
B: Forced to choose 1 and loses!

Known Results
1. Choosing 2 or 3 are bad opening moves for \( A \). In either case, \( B \) can choose the other to guarantee a win. Choosing 4, 6, 8, and 12 are also bad opening moves for \( A \). The sequences (2, 3), (4, 6), (6, 9), and (8, 12) are winning positions for \( B \).
2. Hutching’s Theorem: If \( \gcd(a, b) = 1 \) and \( \{a, b\} \neq \{2, 3\} \), then \( \{a, b\} \) is winning position for \( A \).
3. If \( n \) is prime with \( p \geq 5 \), then \( p \) is a winning position for \( A \).
4. If \( n \) is composite & not equal to 2*3^k, then \( n \) is a winning position for \( B \).

Open Question (John Conway)
If player \( A \) names 16, and both players play optimally thereafter, then who wins? Note that 16 is the smallest number not handled by the facts above.

Simplified Sylver Coinage
In Simplified Sylver Coinage, \( A \) and \( B \) alternate name positive integers from the set \( [n] := \{2, \ldots, n\} \) that are not the sum of nonnegative multiples of previously named numbers among \( [n] \). The player that eliminates the last remaining number is the winner.

Example
Suppose \( n = 10 \). Below is one possible sequence of moves.
\begin{align*}
A: &\quad (4) = (4, 8) \\
B: &\quad (4.5) = (4, 5, 8, 9, 10) \\
A: &\quad (4.5, 6) = (4, 5, 6, 8, 9, 10) \\
B: &\quad (4.5, 6, 3) = (3, 4, 5, 6, 7, 8, 9, 10) \\
A: &\quad (4, 5, 6, 3, 2) = (10), and so \( A \) wins!
\end{align*}