Low Complexity Regularization of Inverse Problems

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Inverse Problems

Recovering \( x_0 \in \mathbb{R}^N \) from noisy observations

\[
y = \Phi x_0 + w \in \mathbb{R}^P
\]

\( \Phi : \mathbb{R}^N \mapsto \mathbb{R}^P \) with \( P \ll N \) (missing information)
Inverse Problems

Recovering $x_0 \in \mathbb{R}^N$ from noisy observations

$$y = \Phi x_0 + w \in \mathbb{R}^P$$

$\Phi : \mathbb{R}^N \mapsto \mathbb{R}^P$ with $P \ll N$ (missing information)

*Examples*: Inpainting, super-resolution, ...
Inverse Problems in Medical Imaging

Tomography projection: \[ \Phi x = \left( p_{\theta_k} \right)_{1 \leq k \leq K} \]
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Magnetic resonance imaging (MRI): \( \Phi x = (\hat{f}(\omega))_{\omega \in \Omega} \)
Inverse Problems in Medical Imaging

Tomography projection: \( \Phi x = (p_{\theta_k})_{1 \leq k \leq K} \)

Magnetic resonance imaging (MRI): \( \Phi x = (\hat{f}(\omega))_{\omega \in \Omega} \)

Other examples: MEG, EEG, ...
Compressed Sensing

\[ \tilde{x}_0 \]

[Rice Univ.]
Compressed Sensing

\[ y[i] = \langle x_0, \varphi_i \rangle \]

\[ P \text{ measures } \ll N \text{ micro-mirrors} \]
Compressed Sensing

\[
y[i] = \langle x_0, \varphi_i \rangle
\]

\(P\) measures \(\ll N\) micro-mirrors
Inverse Problem Regularization

Observations: $y = \Phi x_0 + w \in \mathbb{R}^P$.

Estimator: $x(y)$ depends only on observations $y$ parameter $\lambda$
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Estimator: \( x(y) \) depends only on parameter \( \lambda \)

Example: variational methods

\[
x(y) \in \operatorname{argmin}_{x \in \mathbb{R}^N} \frac{1}{2} \| y - \Phi x \|^2 + \lambda J(x)
\]
Observations: \( y = \Phi x_0 + w \in \mathbb{R}^P. \)

Estimator: \( x(y) \) depends only on \( \lambda \)

Example: variational methods

\[
x(y) \in \arg\min_{x \in \mathbb{R}^N} \frac{1}{2} \| y - \Phi x \|^2 + \lambda \| J(x) \|
\]

Data fidelity

Regularity

Choice of \( \lambda \): tradeoff

\[
\text{Noise level} \quad \| w \| \quad \text{Regularity of } x_0 \quad \| J(x_0) \|
\]
Inverse Problem Regularization

Observations: $y = \Phi x_0 + w \in \mathbb{R}^P$.

Estimator: $x(y)$ depends only on parameter $\lambda$.

Example: variational methods

$\begin{align*}
  x(y) &\in \arg\min_{x \in \mathbb{R}^N} \frac{1}{2} \|y - \Phi x\|^2 + \lambda J(x) \\
        &\text{Data fidelity} \\
        &\text{Regularity}
\end{align*}$

Choice of $\lambda$: tradeoff

No noise: $\lambda \to 0^+$, minimize $x(y) \in \arg\min_{\Phi x = y} J(x)$

Noise level $\|w\|$ 

Regularity of $x_0$ $J(x_0)$
**Inverse Problem Regularization**

*Observations:* \( y = \Phi x_0 + w \in \mathbb{R}^P \).

*Estimator:* \( x(y) \) depends only on parameter \( \lambda \).

*Example:* variational methods

\[
x(y) \in \arg\min_{x \in \mathbb{R}^N} \frac{1}{2} \| y - \Phi x \|^2 + \lambda J(x)
\]

Data fidelity

Regularity

Choice of \( \lambda \): tradeoff

No noise: \( \lambda \rightarrow 0^+ \), minimize \( x(y) \in \arg\min_{\Phi x = y} J(x) \)

Performance analysis:

Criteria on \( (x_0, \|w\|, \lambda) \) to ensure \( \|x(y) - x_0\| = O(\|w\|) \)

---

In this context, the regularity of \( x_0 \) is crucial for ensuring model stability and ensuring that the estimator \( x(y) \) is robust to noise. The choice of \( \lambda \) is a tradeoff between data fidelity and model regularity, which is critical in inverse problems to prevent overfitting to noise in the data.
Overview

• Low-complexity Convex Regularization

• Performance Guarantees: L2 Error

• Performance Guarantees: Model Consistency
Union of models for Data Processing

Union of models: $\mathcal{M} \subset \mathbb{R}^N$ subspaces or manifolds.

Synthesis sparsity:

$\mathcal{M}$

Coefficients $x$  Image $\Psi x$
Union of Models for Data Processing

Union of models: \( \mathcal{M} \subset \mathbb{R}^N \) subspaces or manifolds.

**Synthesis sparsity:**

**Structured sparsity:**

Coefficients \( x \) \( \xrightarrow{\Psi} \) Image \( \Psi x \)
Union of Models for Data Processing

Union of models: $\mathcal{M} \subset \mathbb{R}^N$ subspaces or manifolds.

Synthesis sparsity:

Structured sparsity:

Analysis sparsity:

Coefficients $x$

Image $\Psi x$

Image $x$

Gradient $D^* x$
Multi-spectral imaging:

\[ x_{i,:} = \sum_{j=1}^{r} A_{i,j} S_{j,:} \]
Regularizer: $J : \mathbb{R}^N \rightarrow \mathbb{R}$ convex.

Sub-differential: $\partial J(x) = \{ \eta ; \forall y, J(y) \geq J(x) + \langle \eta, y - x \rangle \}$

*Example:* $J(x) = |x|$. 

![Graph](image)
Regularizer: \( J : \mathbb{R}^N \rightarrow \mathbb{R} \) convex.

Sub-differential: \( \partial J(x) = \{ \eta ; \forall y, J(y) \geq J(x) + \langle \eta, y - x \rangle \} \)

**Example:** \( J(x) = |x| \).

**Example:** \( J(x) = \|x\|_1 = \sum_i |x_i| \).

\[
\partial \|x\|_1 = \left\{ \eta \mid \text{supp}(\eta) = I, \forall j \notin I, |\eta_j| \leq 1 \right\}
\]

\( I = \text{supp}(x) = \{i \mid x_i \neq 0\} \)
Subdifferentials and Linear Models

Regularizer: $J : \mathbb{R}^N \rightarrow \mathbb{R}$ convex.

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Example: $J(x) = \|x\|_1 = \sum_i |x_i|$. 

$\partial \|x\|_1 = \left\{ \eta \setminus \text{supp}(\eta) = I, \forall j \notin I, |\eta_j| \leq 1 \right\}$

$I = \text{supp}(x) = \{i \mid x_i \neq 0\}$

$T_x = \{\eta \setminus \text{supp}(\eta) = I\}$

Linear model: $T_x = \text{VectHull}(\partial J(x))^\perp$
Partly Smooth Functions

\[ J : \mathbb{R}^N \to \mathbb{R} \text{ is partly smooth at } x \text{ for a manifold } \mathcal{M}_x \]

(i) \( J \) is \( C^2 \) along \( \mathcal{M}_x \) around \( x \);

\[ J(x) = \max(0, \|x\| - 1) \]
Partly Smooth Functions

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(i) \( J \) is \( C^2 \) along \( \mathcal{M}_x \) around \( x \);

(ii) \( \text{VecHull}(\partial J(x)) \perp = T_x = \text{Tangent}_x(\mathcal{M}_x) \);

\[ J(x) = \max(0, \|x\| - 1) \]

(ii) \( \iff \forall h \in T_x \perp, \)

\[ t \mapsto J(x + th) \]

is non-smooth at \( t = 0 \).
Partly Smooth Functions

$J : \mathbb{R}^N \to \mathbb{R}$ is partly smooth at $x$ for a manifold $\mathcal{M}_x$

(i) $J$ is $C^2$ along $\mathcal{M}_x$ around $x$;

(ii) $\text{VecHull}(\partial J(x))^\perp = T_x = \text{Tangent}_x(\mathcal{M}_x)$;

(iii) $\partial J$ is continuous on $\mathcal{M}_x$ around $x$.

$J(x) = \max(0, \|x\| - 1)$

(ii) $\iff \forall h \in T_x^\perp$, $t \mapsto J(x + th)$ is non-smooth at $t = 0$. 

[Lewis 2003]
Partly Smooth Functions

$J : \mathbb{R}^N \rightarrow \mathbb{R}$ is partly smooth at $x$ for a manifold $M_x$

(i) $J$ is $C^2$ along $M_x$ around $x$;

(ii) $\text{VecHull}(\partial J(x))^\perp = T_x = \text{Tangent}_x(M_x)$;

(iii) $\partial J$ is continuous on $M_x$ around $x$.

$J(x) = \max(0, \|x\| - 1)$

(ii) $\iff \forall h \in T_x^\perp,$

$t \mapsto J(x + th)$

is non-smooth at $t = 0$.

**Important:** in general $M_x \neq T_x$
Examples of Partly-smooth Regularizers

\( \ell^1 \) sparsity: \( J(x) = \|x\|_1 \)

\( \mathcal{M}_x = T_x = \{z ; \text{supp}(z) \subset \text{supp}(x)\} \)
Examples of Partly-smooth Regularizers

\( \ell^1 \) sparsity: \( J(x) = \|x\|_1 \) \( M_x = T_x = \{ z ; \text{supp}(z) \subset \text{supp}(x) \} \)

Structured sparsity: \( J(x) = \sum_b \|x_b\| \) same \( M_x \)

\( J(x) = \|x\|_1 \) \( M_x \) \( J(x) = |x_1| + \|x_{2,3}\| \)
Examples of Partly-smooth Regularizers

\( \ell^1 \) sparsity: \( J(x) = \| x \|_1 \) \quad \mathcal{M}_x = T_x = \{ z ; \text{supp}(z) \subset \text{supp}(x) \} \\

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Nuclear norm: \( J(x) = \| x \|_* \) \quad \mathcal{M}_x = \{ x ; \text{rank}(z) = \text{rank}(x) \} 

\[ J(x) = \| x \|_1 \quad \mathcal{M}_x \quad J(x) = | x_1 | + \| x_{2,3} \| \\
J(x) = \| x \|_* \]
Examples of Partly-smooth Regularizers

\( l^1 \) \( \text{sparsity: } J(x) = \|x\|_1 \quad \mathcal{M}_x = T_x = \{z \mid \text{supp}(z) \subset \text{supp}(x)\} \)

\( \text{Structured sparsity: } J(x) = \sum_b \|x_b\| \quad \text{same } \mathcal{M}_x \)

\( \text{Nuclear norm: } J(x) = \|x\|_* \quad \mathcal{M}_x = \{x \mid \text{rank}(z) = \text{rank}(x)\} \)

\( \text{Anti-sparsity: } J(x) = \|x\|_\infty \quad \mathcal{M}_x = T_x = \{z \mid z_I \propto x_I\} \)

\( I = \{i \mid |x_i| = \|x\|_\infty\} \)
Overview

• Low-complexity Convex Regularization

• Performance Guarantees: L2 Error

• Performance Guarantees: Model Consistency
Dual Certificates

Noiseless recovery: \[ \min_{\Phi x = \Phi x_0} J(x) \quad (\mathcal{P}_0) \]
**Noiseless recovery:** 
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**Proposition:**
\[ x_0 \text{ solution of } (\mathcal{P}_0) \iff \exists \eta \in \mathcal{D}(x_0) \]

**Dual certificates:**
\[ \mathcal{D}(x_0) = \text{Im}(\Phi^*) \cap \partial J(x_0) \]
Dual Certificates

**Noiseless recovery:**
\[ \min_{\Phi x = \Phi x_0} J(x) \quad (P_0) \]

**Proposition:**
\[ x_0 \text{ solution of } (P_0) \iff \exists \eta \in D(x_0) \]

**Dual certificates:**
\[ D(x_0) = \text{Im}(\Phi^*) \cap \partial J(x_0) \]

**Example:**
\[ J(x) = \|x\|_1 \quad \Phi x = x \star \varphi \]
\[ D(x_0) = \{ \eta = x \star \varphi ; \eta_i = \text{sign}(x_{0,i}) , \|\eta\|_{\infty} \leq 1 \} \]
Non degenerate dual certificate:
\[ \bar{D}(x_0) = \text{Im} (\Phi^*) \cap \text{ri}(\partial J(x_0)) \]

\( \text{ri}(E) = \) relative interior of \( E \)
\( = \) interior for the topology of \( \text{aff}(E) \)
Non degenerate dual certificate:

\[ \tilde{\mathcal{D}}(x_0) = \text{Im}(\Phi^*) \cap \text{ri}(\partial J(x_0)) \]

\( \text{ri}(E) = \) relative interior of \( E \)

\( = \) interior for the topology of \( \text{aff}(E) \)

**Theorem:** [Fadili et al. 2013]

If \( \exists \eta \in \tilde{\mathcal{D}}(x_0) \), for \( \lambda \sim \|w\| \) one has \( \|x^* - x_0\| = O(\|w\|) \)
Dual Certificates and L2 Stability

Non degenerate dual certificate:
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[Grassmair, Haltmeier, Scherzer 2010]: \( J = \| \cdot \|_1 \).
[Grassmair 2012]: \( J(x^* - x_0) = O(\| w \|) \).
Non degenerate dual certificate:
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\[ x^* = \Phi x = \Phi x_0 \]

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[Grassmair 2012]: \( J(x^* - x_0) = O(\| w \|) \).

\( \longrightarrow \) The constants depend on \( N \) ...
Compressed Sensing Setting

Random matrix: \( \Phi \in \mathbb{R}^{P \times N}, \Phi_{i,j} \sim \mathcal{N}(0,1), \) i.i.d.
Compressed Sensing Setting

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Sparse vectors: \( J = \| \cdot \|_1 \).

**Theorem:** Let \( s = \| x_0 \|_0 \). If
\[
P \geq 2s \log \left( \frac{N}{s} \right)
\]
Then \( \exists \eta \in \tilde{D}(x_0) \) with high probability on \( \Phi \).

[Chandrasekaran et al. 2011] [Rudelson, Vershynin 2006]
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Then $\exists \eta \in \mathcal{D}(x_0)$ with high probability on $\Phi$.

Low-rank matrices: $J = \| \cdot \|_*$.

**Theorem:** Let $r = \text{rank}(x_0)$. If

$$P \geq 3r(N_1 + N_2 - r)$$

Then $\exists \eta \in \mathcal{D}(x_0)$ with high probability on $\Phi$. 

[Chandrasekaran et al. 2011]

[Chandrasekaran et al. 2011]

[Rudelson, Vershynin 2006]
### Compressed Sensing Setting

**Random matrix:** \( \Phi \in \mathbb{R}^{P \times N}, \quad \Phi_{i,j} \sim \mathcal{N}(0, 1), \text{ i.i.d.} \)

**Sparse vectors:** \( J = \| \cdot \|_1. \)

**Theorem:** Let \( s = \| x_0 \|_0. \) If
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**Low-rank matrices:** \( J = \| \cdot \|_* \).

**Theorem:** Let \( r = \text{rank}(x_0). \) If
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\( \longrightarrow \) Similar results for \( \| \cdot \|_{1,2}, \| \cdot \|_{\infty}. \)

*References:
- [Rudelson, Vershynin 2006]
- [Chandrasekaran et al. 2011]*
Although Theorem \( \| \cdot \|_1 \)
vectors, and we aim to extract the two constituents from the mixture. More precisely, suppose that we measure

Suppose that in a sparse vector in the theoretical and empirical phase transitions. Figure 2.4.

To solve the demixing problem, we describe a convex programming technique proposed in \[ \| \cdot \|_1 \] for identifying a low-rank matrix. In each panel, the colormap indicates the empirical probability of success (black = 0%; white = 100%). The yellow curve marks the theoretical prediction of the phase transition from Theorem 2.6.

For more information, see \[ \| \cdot \|_1 \] and \[ \| \cdot \|_* \] from Sections 2.2.

From [Amelunxen et al. 2013]
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- Low-complexity Convex Regularization

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- Performance Guarantees: Model Consistency
Minimal-norm certificate:
\[ \eta_0 = \arg\min_{\eta = \Phi^* q \in \partial J(x_0)} \| q \| \]
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\[ \eta_0 = \arg\min_{\eta=\Phi^* q \in \partial J(x_0)} \|q\| \]

\[ \partial J(x_0) \subset A(x_0) = \text{AffHull}(\partial J(x_0)) \]

Case \( J = \| \cdot \|_1 \)
**Minimal Norm Certificate**

**Minimal-norm certificate:**
\[ \eta_0 = \underset{\eta = \Phi^* q \in \partial J(x_0)}{\text{argmin}} \| q \| \]

\[ \partial J(x_0) \subset A(x_0) = \text{AffHull}(\partial J(x_0)) \]

**Linearized pre-certificate:**
\[ \eta_F = \underset{\eta = \Phi^* q \in A(x_0)}{\text{argmin}} \| q \| \]

Case \( J = \| \cdot \|_1 \)

\[ T = T_{x_0} \]

\[ \partial J(x_0) \]

\[ A(x_0) \]
Minimal-norm certificate:
\[ \eta_0 = \arg \min_{\eta = \Phi^* q \in \partial J(x_0)} \|q\| \]

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Linearized pre-certificate:
\[ \eta_F = \arg \min_{\eta = \Phi^* q \in A(x_0)} \|q\| \]

\[ \rightarrow \eta_F \text{ is computed by solving a linear system.} \]

\[ \rightarrow \text{One does not always have } \eta_F \in D(x_0) ! \]
Minimal-norm certificate:
\[ \eta_0 = \arg\min_{\eta = \Phi^* q \in \partial J(x_0)} \| q \| \]

Thus \( \partial J(x_0) \subset A(x_0) = \text{AffHull}(\partial J(x_0)) \)

Linearized pre-certificate:
\[ \eta_F = \arg\min_{\eta = \Phi^* q \in A(x_0)} \| q \| \]

\( \eta_F \) is computed by solving a linear system.

\( \eta_F \) does not always belong to \( D(x_0) \).!
**Theorem:** If $\eta_F \in \bar{D}(x_0)$, there exists $C$ such that if

$$\max (\lambda, \|w\|/\lambda) \leq C$$

the unique solution $x^*$ of $P_\lambda(y)$ for $y = \Phi x_0 + w$ satisfies

$$x^* \in M_{x_0} \quad \text{and} \quad \|x^* - x_0\| = O(\|w\|, \lambda)$$

[Vaiter et al. 2014]
Theorem: If \( \eta_F \in \bar{D}(x_0) \), there exists \( C \) such that if

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\max (\lambda, \| w \| / \lambda) \leq C
\]

the unique solution \( x^* \) of \( \mathcal{P}_\lambda(y) \) for \( y = \Phi x_0 + w \) satisfies

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x^* \in \mathcal{M}_{x_0} \quad \text{and} \quad \| x^* - x_0 \| = O(\| w \|, \lambda)
\]

Previous works:

[Fuchs 2004]: \( J = \| \cdot \|_1 \).

[Bach 2008]: \( J = \| \cdot \|_{1,2} \) and \( J = \| \cdot \|_* \).

[Vaiter et al. 2011]: \( J = \| D^* \cdot \|_1 \).
Compressed Sensing Setting

Random matrix: \( \Phi \in \mathbb{R}^{P \times N}, \quad \Phi_{i,j} \sim \mathcal{N}(0, 1), \) i.i.d.

Sparse vectors: \( J = \| \cdot \|_1. \)

**Theorem:** Let \( s = \| x_0 \|_0. \) If

\[
P > 2s \log(N)
\]

Then \( \eta_0 \in \bar{D}(x_0) \) with high probability on \( \Phi. \)

[Wainwright 2009]
[Dossal et al. 2011]
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Phase transitions:

- **L^2 stability**
  - \( P \sim 2s \log(N/s) \)

vs.

Model stability

- \( P \sim 2s \log(N) \)

[Wainwright 2009]
[Dossal et al. 2011]
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Random matrix: \( \Phi \in \mathbb{R}^{P \times N} \), \( \Phi_{i,j} \sim \mathcal{N}(0, 1) \), i.i.d.

Sparse vectors: \( J = \| \cdot \|_1 \).

**Theorem:** Let \( s = \| x_0 \|_0 \). If

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P > 2s \log(N)
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Then \( \eta_0 \in \widetilde{D}(x_0) \) with high probability on \( \Phi \).

Phase transitions:

- \( L^2 \) stability: \( P \sim 2s \log(N/s) \)
- Model stability: \( P \sim 2s \log(N) \)

\( \rightarrow \) Similar results for \( \| \cdot \|_1,2, \| \cdot \|_*, \| \cdot \|_\infty \).
Compressed Sensing Setting

Random matrix: \( \Phi \in \mathbb{R}^{P \times N}, \; \Phi_{i,j} \sim \mathcal{N}(0,1), \) i.i.d.

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**Theorem:** Let \( s = \| x_0 \|_0. \) If

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Then \( \eta_0 \in \tilde{D}(x_0) \) with high probability on \( \Phi. \)

---

Phase transitions: \( L^2 \) stability \( P \sim 2s \log(N/s) \) vs. Model stability \( P \sim 2s \log(N) \)

\[\rightarrow\] Similar results for \( \| \cdot \|_{1,2}, \| \cdot \|_*, \| \cdot \|_\infty. \)

\[\rightarrow\] Not using RIP technics (non-uniform result on \( x_0). \)
$\Phi x = \sum_i x_i \varphi(\cdot - \Delta i)$

$J(x) = \|x\|_1$

Increasing $\Delta$:

$\rightarrow$ reduces correlation.

$\rightarrow$ reduces resolution.
\[ \Phi x = \sum_i x_i \varphi(\cdot - \Delta i) \]

\[ J(x) = \|x\|_1 \]

Increasing \( \Delta \):
- reduces correlation.
- reduces resolution.

\[ I = \{ j \mid x_0(j) \neq 0 \} \]
\[ \|\eta_{F,I^c}\|_\infty < 1 \]

\[ \eta_0 = \eta_F \in \bar{D}(x_0) \]

support recovery.
Conclusion

*Partial smoothness:* encodes models using singularities.
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Performance measures $L^2$ error model $\rightarrow$ different CS guarantees
Conclusion

**Partial smoothness**: encodes models using singularities.

Performance measures $L^2$ error $\models$ different CS guarantees

Specific certificate: $\eta_0, \eta_F, \cdots$
Performance measures $L^2$ error model different CS guarantees

Partial smoothness: encodes models using singularities.

Open problems:
- CS performance with arbitrary gauges.
- Infinite dimensional regularizations (BV, ...)
- Convergence discrete $\rightarrow$ continuous.