A Microlocal Viewpoint of Rendering

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The radiosity equation

\[ R(x) = E(x) + \int_{\Gamma} K(x, y)R(y)\,dy \quad K(x, y) \overset{\text{def.}}{=} \rho(x)V(x, y)\frac{\cos(\theta_x)\cos(\theta_y)}{\|x - y\|^2} \]

Radiosity equation: \((\text{Id} - K)R = E\).

- **\(R(x)\):** radiosity
- **\(E(x)\):** emission.
- **\(V(x, y)\):** visibility between \(x \leftrightarrow y\).
- **\(\rho(x)\):** reflectivity

- Setting: operator \(K\) with curvilinear singularities.
- Goal: represent efficiently the singularities of \(K\).
- Singularities of \(K \leftrightarrow \) geometry of \(\mathcal{M}\).

Proposal: use microlocal analysis to study and represent \(K\).
Flatland Example

\[ |\rho| < 1 \implies \|K\| < 1 \]

\[ R = \sum_n K^n E \]
More Complex Example

Kernel $K(x, y)$

Visibility $V(x, y)$

$E$

$KE$

$K^2E$

$K^3E$
Examples in 3D

First shadow $KE$

Radiosity $R$
**Microlocal Description of Light/Shadow**

**Wavefront set:**
\[ \text{WF}(u) \overset{\text{def.}}{=} \{(x, \xi) \in T^*S_x \setminus \hat{u}_x(\xi) \text{ decays slowly with } |\xi|\} \]

**Singular support:**
\[ S(u) \overset{\text{def.}}{=} \{x \setminus (x, \theta) \in WS(u)\} \]

**Light source:** \( u = 1_\Omega \)

\[ S(u) = \partial \Omega \quad \xi \perp \partial \Omega \]

**Light-shadow relation (LSR) \( \mathcal{K} \):**
\[ \mathcal{K} = WS(K) \subset T^*(S_x \times S_y) \]

Wavefront set of \( K(x, y) \) for \( (x, y) \in S_x \times S_y \).
Gemmetrical Description of LSR

Light at infinity
\( \omega \overset{\text{def.}}{=} x \)

\( \omega = x \) at infinity

Local light
\( \omega(p) \overset{\text{def.}}{=} x - p \)

\( \omega = p - x \)

Microray \( (x, p, \xi), p \in \mathcal{M} \) and \( (x, \xi) \in T^*S_x \).

Microlocally grazing: \( \omega \in T_p\mathcal{M} \) and \( \omega \times \xi \in T_p\mathcal{M} \)

Implicit equation: \( \langle n_p, \omega(p) \rangle = 0 \) and \( \langle n_p, \omega(p) \times \xi \rangle = 0 \)
Geometrical Theorem

Theorem: Let $\mathcal{O}$ be an open neighborhood of $S_x \times S_y$ without self-occlusion. A LS relation $(x, \xi, y, \eta)$ in $\mathcal{K} \cup \mathcal{O}$ generates microlocally grazing rays $(x, p, \xi)$ and $(y, p, \eta)$.

Globally $\mathcal{K}$ is not a graph of a function.
Multiple associations $(x, \xi) \mapsto \{(y, \eta), (y', \eta'), \ldots\}$. 
Composition of the LSR

LSR composition:

\[(y, \eta) \in \mathcal{K} \circ WF(u) \iff \exists (x, \xi) \in WF(u) \text{ such that } (x, \xi, \eta, y) \in \mathcal{K}\]

\[\text{Point light: } u = \delta_x\]

\[\text{Extended light: } u = 1_{\Omega}\]

*Theorem:* The \(n^{th}\) shadow wavefront \(WF(K^n E)\) satisfies \(WF(K^{n+1} E) \subset \mathcal{K} \circ WF(K^n E)\).
Point Light: Silhouette

Silhouette: $\text{Sil}(\omega) = \{ p \in \mathcal{M} \setminus \langle \omega(p), n_p \rangle = 0 \}$.
Silhouette Regularity

Silhouette: Sil(\(\omega\)) = \{p \in \mathcal{M} \mid \langle \omega(p), n_p \rangle = 0 \}.

**Theorem:** For a smooth manifold \(\mathcal{M}\), Sil(\(\omega\)) is a smooth curve away from points where \(d n_p(\omega(p)) = 0\).

“parabolic points”
Silhouette Examples

Sil(\(\omega\)) regular

Sil(\(\omega\)) singular

Shadow boundary \(S(KE)\) singular

Shadow boundary \(S(KE)\) singular
Extended Light Source

\[ f_\omega(p) = \langle n_p, \omega \rangle = 0 \quad \text{and} \quad g_{\omega,\xi}(p) = \langle n_p, \omega \times \xi \rangle = 0 \]
Extended Light: Regularity of the LSR

Implicit formulation:

\[ f_\omega (p) = \langle n_p, \omega \rangle = 0 \quad \text{and} \quad g_\omega,\xi (p) = \langle n_p, \omega \times \xi \rangle = 0 \]

\[ df_\omega (p) = dn_p(\omega) \quad \text{and} \quad dg_{\omega,\xi} (p) = dn_p(\omega \times \xi) \]

\[ \text{Rank}(df_\omega (p), dg_{\omega,\xi} (p)) = 2 \iff \det(I_p) \neq 0 \]

Theorem: Away from \((\omega, \xi)\)-parabolic points, the LSR \( \mathcal{K} \) is locally the graph of a function

\[(x, \xi) \in T^*S_x \mapsto p \in \mathcal{M} \mapsto (y, \eta) \in T^*S_x\]
LSR Precomputation

Precomputed data:
\[(x_i, \xi_j) \mapsto \{p^1_{i,j}, p^2_{i,j}, \ldots, \}\]

ODE interpolation:
- point light:
  \[\frac{dp(t)}{dt} = \pm \omega_{p(t)} \times n_{p(t)}\]
- extended light: more complex ODE …

Grazing rays inverse sampling:
- Traverse normals \(n_p\) sampled on the surface.
- Associate each normal with \((x_i, \xi_j)\) or \((\omega_i, \xi_j)\) on the bounding sphere(s).

\rightarrow Light at infinity: match \(\xi_j \approx n_p\)
\rightarrow Local light: match \(\langle x_i - p, n_p \rangle \approx 0\) and \(\langle (x_i - p) \times \xi_j, n_p \rangle \approx 0\).
Numerical Example

Geometry image

\[ \tau \in [0, 1]^2 \leftrightarrow p(\tau) \in \mathbb{R}^3 \]

Silhouette in \( \mathbb{R}^3 \)

Silhouette over \( \tau \) domain

Projected silhouette

Sing. support \( S(K1_\Omega) \)
Numerical Example
Conclusion

- Microlocal formalism: describe the singularity of the radiosity kernel.
- Geometrical interpretation: away from exceptional points, no singularity in phase space.
- Precomputations: sparse sampling of the LSR + local ODE integration.
- Extension: handle self occlusions.
  \[\Rightarrow\] Prune the shadow boundary.
- Application: fast shadow rendering by approximating $KE$ and $K^2E$.
- Importance sampling monte carlo simulation of light transport.